p-adic approach to differential equations

Daniel Vargas-Montoya, Masha Vlasenko

October 2024

This is a first draft of notes of a short lecture course given by the authors during the workshop *Algebraicity and transcendence for singular differential equations* at Erwin Shrödinger Institute in Vienna on 7-10 October 2024.

Contents

1 p-adic Frobenius structure for ordinary differential equations

1.1 Equivalence of differential systems

Let $K \supset \mathbb{C}(t)$ be a differential field. That is, a field with the derivation $\frac{d}{dt} : K \to K$ which extends the usual derivation on the field of rational functions $\mathbb{C}(t)$. Take two matrices $A, B \in \mathbb{C}(t)^{n\times n}$ and consider linear differential systems

$$(I) \quad \frac{dU}{dt} = AU \qquad \text{and} \qquad (II) \quad \frac{dV}{dt} = BV.$$

Definition 1. *We say that* (II) is equivalent to (I) over K *if there exists a matrix* $H \in GL_n(K)$ *such that*

$$\frac{dH}{dt} = AH - HB. \tag{1}$$

Notation: (2) $\sim^K$ *(1).*

Note that this relation is symmetric because H^{-1} will satsfy (1) with the roles of A and B interchanged. We first make formal algebraic observations about the meaning of this differential equation:

(i) If V is a vector solution to (2) with entries in a possibly bigger field, then $U = HV$ is a vector solution to (1):

$$\frac{dU}{dt} = \frac{d}{dt}(HV) = (AH - HB)V + HBV = AHV = AU.$$

(ii) If U, V are fundamental matrices of solutions to (1) and (2) respectively, then $\Lambda = U^{-1}HV$ is a constant matrix:

$$\frac{d}{dt}\left(U^{-1}HV\right) = (-U^{-1}A)HV + U^{-1}(AH - HB)V + U^{-1}HBV = 0.$$

Example 2 (local study / Fuchsian theory of regular singularities)**.** *Let* $A \in \mathbb{C}(t)^{n\times n}$ *with no pole at* $t = 0$*. Let* $\mathcal{O}$ *be the ring of germs of holomorphic functions near* $t = 0$ *and* $K = \mathcal{O}[t^{-1}]$ *be the field of germs of meromorphic functions. Then there exists a constant matrix* $\Gamma \in \mathbb{C}^{n\times n}$ *such that*

$$\frac{dU}{dt} = \frac{A(t)}{t}U \qquad \sim^K \qquad \frac{dV}{dt} = \frac{\Gamma}{t}V.$$

Differential systems of this kind are either regular or regular singular at $t = 0$*, which means that their solutions have moderate growth on approach to this point (see [8, Theorem 1.3.1]). Note that* $V = t^{\Gamma}$ *is a fundamental solution matrix of the second system and* $M_0 = \exp(2\pi i\Gamma)$ *is its monodromy around* $t = 0$*.Since elemnts of* K *are single-valued at* $t = 0$ *(have trivial monodromy), local monodromy matrices of two equivalent systems are conjugate by an element of* $GL_n(\mathbb{C})$*. Two such systems of this kind (regular or regular singular) are equivalent over* K *if and only if their local monodromy matrices* M_0 *are conjugate by an element of* $GL_n(\mathbb{C})$ *([8, Corollary 1.3.2]).*

1.2 p-adic analytic elements

Let p be a prime number. The Gauss norm on $\mathbb{Q}[t]$ is defined as

$$|a_0 + a_1 t + \ldots + a_n t^n|_{\mathcal{G}} = \max_{0\le i\le n} |a_i|_p.$$

It satisfies the properties

- $|f + g|_{\mathcal{G}} \le \max(|f|_{\mathcal{G}}, |g|_{\mathcal{G}})$ (non-Archimedean triangle inequality);
- $|f \cdot g|_{\mathcal{G}} = |f|_{\mathcal{G}} \cdot |g|_{\mathcal{G}}$ (Gauss' lemma).

This non-Archimedean norm extends uniquely to the field of rational functions $\mathbb{Q}(t)$ preserving the properties (i)-(ii). In particular, for a ratio of two polynomials one has

$$\left|\frac{\sum_i a_i t^i}{\sum_j b_j t^j}\right|_{\mathcal{G}} = \frac{\max_i |a_i|_p}{\max_j |b_j|_p}.$$

With the Gauss norm $\mathbb{Q}(t)$ becomes an incomplete discretely valued field.

Definition 3. *The field of* p*-adic analytic elements* E_p *is the completion of* $\mathbb{Q}(t)$ *with respect to the Gauss norm.*

Elements of E_p are p-adic limits of rational functions. One class of examples is given by series $\sum_{n=0}^{\infty} a_n t^n$ with $|a_n|_p \to 0$ as $n \to \infty$. We will encounter more sophisticated examples below.

Proposition 4. *The following operations on* $\mathbb{Q}(t)$ *are continuous with respect to the Gauss norm:*

(i) Frobenius endomorphism $f(t) \mapsto f(t^p)$*,*

(ii) derivation $\frac{d}{dt}$*.*

Proof. Property $|f(t^p)|_{\mathcal{G}} = |f(t)|_{\mathcal{G}}$ follows immediately from the definition of the Gauss norm. For (ii) we note that for $f = \sum_i a_i t^i \in \mathbb{Q}[t]$ one has $|f'|_{\mathcal{G}} = \max_i |i\, a_i|_p \leq \max_i |a_i|_p = |f|_{\mathcal{G}}$. With this we can make the conclusion for the ratio of two polynomials:

$$\Big|\frac{d}{dt}\left(\frac{f}{g}\right)\Big|_{\mathcal{G}} = \Big|\frac{f'g - g'f}{g^2}\Big|_{\mathcal{G}} = \frac{|f'g - g'f|_{\mathcal{G}}}{|g|^2_{\mathcal{G}}} \leq \frac{\max(|f'g|_{\mathcal{G}}, |g'f|_{\mathcal{G}})}{|g|^2_{\mathcal{G}}} \leq \frac{|f|_{\mathcal{G}} \cdot |g|_{\mathcal{G}}}{|g|^2_{\mathcal{G}}} = \Big|\frac{f}{g}\Big|_{\mathcal{G}}.$$

□

Hence we can conclude that both Frobenius endomorphism and derivation extend to the field E_p.

1.3 p-adic Frobenius structure

Let $A \in \mathbb{Q}(t)^{n\times n}$. Observe that if $U(t)$ is a solution to $\frac{dU}{dt} = AU$ then $V(t) = U(t^{p^h})$ is a solution of $\frac{dV}{dt} = p^h t^{p^h-1} A(t^{p^h})V$.

Definition 5. *A p-adic Frobenius structure of period h for the differential system* $\frac{dU}{dt} = AU$ *is a matrix* $\Phi \in GL_n(E_p)$ *satisfying the differential equation*

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t) - p^h t^{p^h-1}\Phi(t)A(t^{p^h}). \tag{2}$$

When $h = 1$ we simply call Φ a p-adic Frobenius structure

Example 6. *Consider* $\frac{dU}{dt} = \frac{1}{2}\frac{1}{1-t}U$*. The unique solution is given by* $U(t) = \frac{1}{\sqrt{1-t}}$*. In the view of property (ii) from § 1.1, existence of a p-adic Frobenius structure for a system of rank 1 is equivalent to the fact that* $\Phi(t) = U(t)/U(t^p)$ *is a p-adic analytic element. Let us check that this is indeed the case for our system when* $p \neq 2$*. We first perform a formal computation:*

$$\frac{U(t)}{U(t^p)} = \sqrt{\frac{1-t^p}{1-t}} = (1-t)^{\frac{p-1}{2}}\sqrt{\frac{1-t^p}{(1-t)^p}} = (1-t)^{\frac{p-1}{2}}\left(1 + \frac{p\,g(t)}{(1-t)^p}\right)^{1/2} \quad \text{with} \quad g(t) = \frac{1-t^p-(1-t)^p}{p}$$

$$= (1-t)^{\frac{p-1}{2}}\sum_{k=0}^{\infty}\binom{1/2}{k}p^k\frac{g(t)^k}{(1-t)^{p\,k}}.$$

Here for $k \geq 2$ *we have*

$$\binom{1/2}{k} = \frac{(1/2)(1/2-1)\dots(1/2-(k-1))}{k!} = (-1)^{k-1}\frac{1\cdot 3\cdot \ldots \cdot (2k-3)}{2^k k!}$$

$$= (-1)^{k-1}\frac{(2k-3)!}{2^{2k-2}k!(k-2)!} = (-1)^{k-1}\frac{1}{2^{2k-2}(2k-2)}\binom{2k-2}{k},$$

from which it clearly follows that the p-adic valuation of $\binom{1/2}{k}p^k$ *grows infinitely as* $k \to \infty$*. Thus the partial sums of the series representation we computed above will give a Cauchy sequence with respect to the Gauss norm. It limit will represent* $\Phi(t) = U(t)/U(t^p)$ *as an element of* E_p*.*

Now we would like to mention several facts about p-adic convergence of solutions of differential systems. The reader may prove the following proposition as an exercise, or consult [19] where this fact is proved in a more general context.

Proposition 7 (p-adic Cauchy theorem, Élisabeth Lutz)**.** *Suppose the entries of* $A \in \mathbb{Q}_p[\![t]\!]^{n\times n}$ *have a positive radius of p-adic convergence. Then there exists an invertible matrix* $U \in GL_n(\mathbb{Q}_p[\![t]\!])$ *such that* $\frac{dU}{dt} = AU$*. This matrix is unique up to multiplication from the right by constant invertible matrices* $C \in GL_n(\mathbb{Q}_p)$ *and entries of U have a positive radius of p-adic convergence.*

Existence of a p-adic Frobenius structure implies that the radius of p-adic convergence of solutions is at least 1:

Theorem 8 (Dwork). *If $A \in \mathbb{Q}(t)^{n\times n}$ has no poles in the p-adic disk $|t|_p < 1$ and has a Frobenius structure, then the fundamental matrix of solutions to $\frac{dU}{dt} = AU$, $U \in \mathbb{Q}_p[\![t]\!]^{n\times n}$, also converges for $|t|_p < 1$.*

Here is a negative example: the rank 1 differential system $\frac{dU}{dt} = U$ has no p-adic Frobenius structure for any prime p. This fact follows from the above theorem because the solution $U(t) = \exp(t)$ has radius of p-adic convergence $p^{-\frac{1}{p-1}} < 1$. One can also give a direct argument, not involving Dwork's theorem. Instead, demonstrate that $\exp(t - t^p)$ is not a p-adic analytic element. See exercise X below.
In the situation of Proposition 7 we can conclude from (ii) of § 1.1 that the differential equation (2) defining the Frobenius structure has n^2-dimensional $\mathbb{Q}_p$-vector space of solutions $\Phi \in \mathbb{Q}_p[\![t]\!]^{n\times n}$ given by $\Phi(t) = U(t)\Lambda U(t^{p^h})^{-1}$ with any $\Lambda \in \mathbb{Q}_p^{n\times n}$. Their entries have a positive radius of p-adic convergence, and we can ask for which Λ we actually get entries in E_p. The following theorem tells us that if such Λ exists it is unique up to a scalar multiple.

Theorem 9 (Dwork, [14]). *Let $A \in \mathbb{Q}(t)^{n\times n}$ and suppose that the differential system $\frac{dU}{dt} = AU$ satisfies the following properties:*

- *all its singularities are regular,*
- *all local exponents are in $\mathbb{Q} \cap \mathbb{Z}_p$,*
- *the difference of any two singularities has p-adic valuation 0,*
- *it is irreducible over $\mathbb{Q}_p(t)$.*

Then if a p-adic Frobenius structure exists, it is unique up to multiplication by a non-zero constant.

In these lectures, we will discuss the existence of a p-adic Frobenius structure and its arithmetic consequences.

1.4 The case of differential equations

For a monic linear differential operator

$$L = (\frac{d}{dt})^n + a_1(t)(\frac{d}{dt})^{n-1} + \ldots + a_n(t) \in \mathbb{Q}(t)[\frac{d}{dt}]$$

its *companion matrix* is defined as

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ & \vdots & & \ldots \\ -a_n & -a_{n-1} & & \end{pmatrix}.$$

Then vector solutions to the differential system $\frac{dU}{dt} = AU$ are precisely of the form

$$U(t) = (y(t), y'(t), \ldots, y^{(n-1)}(t))^T,$$

where $y(t)$ is a solution to $Ly = 0$. We would like to consider the cases when L is regular at $t = 0$ (so all $a_i(t)$ are analytic at $t = 0$) or $t = 0$ is a regular singularity (which happens when for each i the coefficient $a_i(t)$ has a pole of order at most i at $t = 0$, see [7]). Both for the analysis of singularity at $t = 0$ and for describing the Frobenius structure near this point it is convenient to rewrite the differential equation in terms of the derivation $\theta = t\frac{d}{dt}$. Multuplying our operator on the left by t^n and using formula $t^i(d/dt)^i = \theta(\theta+1)\ldots(\theta+i-1)$ we may assume that

$$L = \theta^n + b_1(t)\theta^{n-1} + \ldots + b_n(t)$$

with all b_i analytic at $t=0$. Recall that local exponents at $t=0$ are the roots of the indicial polynomial $X^n + b_1(0)X^{n-1} + \ldots + b_{n-1}(0)X + b_n$. The companion matrix will be

$$B(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ & \vdots & & \ldots \\ -b_n & -b_{n-1} & & \end{pmatrix}$$

and solutions to $\theta U = BU$ are of the form $U(t) = (y, \theta y, \ldots, \theta^{n-1} y)^T$.
Denote $q = p^h$ and consider the operator

$$L^{(q)} = \sum_{i=0}^{n} b_i(t^q) q^i \theta^{n-i}.$$

Since $\theta^i(y(t^q)) = q^i(\theta^i y)(t^q)$, this is the operator whose solutions are given by $y(t^q)$ where $y(t)$ is a solution to L. Note that $L^{(q)}$ has regular singularity at $t=0$ with indicial polynomial $\sum_{i=0}^n b_i(0) q^i X^{n-i}$, and hence its local exponents are q-multiples of the local exponents of L. The equation for the Frobenius structure of period h now transforms into

$$\theta\Phi(t) = B(t)\Phi(t) - q\Phi(t)B(t^q). \tag{3}$$

Assume that the local exponents of L at $t=0$ are rational and let d be the least common multiple of their denominators. We denote by

$$Sol(L) \subset \mathbb{Q}_p[\![t]\!][t^{-1/d}, \log(t)]$$

the n-dimensional $\mathbb{Q}_p$-vector space generated by solutions of L. Similarly, we have the $\mathbb{Q}_p$-vector subspace $Sol(L^{(q)})$ and the isomorphism $Sol(L) \to Sol(L^{(q)})$ given by $t \mapsto t^p$ and $\log(t) \mapsto p\log(t)$.

Proposition 10. *Let L be a differential operator of order n and $t=0$ is a regular singularity of L with rational local exponents. The following conditions are equivalent:*

- *(i) There exists a solution Φ to (3) with entries $\Phi_{ij} \in E_p \cap \mathbb{Q}_p[\![t]\!]$.*
- *(ii) There exists an invertible linear map $\mathcal{A} : Sol(L^{(q)}) \to Sol(L)$ given by a differential operator $\mathcal{A} = \sum_{i=0}^{n-1} A_i(t)\theta^i$ with coefficients $A_i \in E_p \cap \mathbb{Q}_p[\![t]\!]$.*

Proof. (i)⇒(ii) Let y be a solution to $L(y) = 0$ and $U(t) = (y, \theta y, \ldots, \theta^{n-1}y)^T$. Then $\Phi(t)U(t^q) = (\tilde{y}, \theta\tilde{y}, \ldots, \theta^{n-1}\tilde{y})^T$ for some solution $\tilde{y} \in Sol(L)$. Here

$$\tilde{y}(t) = \sum_{j=0}^{n-1} \Phi_{oj}(t)(\theta^j y)(t^q) = \sum_{j=0}^{n-1} \Phi_{oj}(t) q^{-j} \theta^j (y(t^q)),$$

and hence we obtain a differential operator between the spaces of solutions

$$\mathcal{A} = \sum_{j=0}^{n-1} q^{-j} \Phi_{0,j}\, \theta^j : Sol(L^{(q)}) \to Sol(L).$$

To show that this operator is invertible we choose any basis $y_0, \ldots, y_{n-1} \in Sol(L)$. Then $y_i(t^p)$, $0 \le i \le n-1$ is a basis in $Sol(L^{(q)}$. Let $U = (\theta^i y_j)_{0 \le i,j \le n-1}$ and let $\tilde{U} = (\theta^i \tilde{y}_j)_{0 \le i,j \le n-1}$ be a similar Wronskian matrix for the images $\tilde{y}_i = \mathcal{A}(y_i(t^q))$. Since $\tilde{U}(t) = \Phi(t)U(t^q)$, we obtain that the Wronskian determinant of the images in non-zero

$$W(\tilde{y}_0, \ldots, \tilde{y}_{n-1}) = \det(\tilde{U}) = \det\Phi \cdot \det U \neq 0,$$

and hence these solutions are linearly independent.
(ii)⇒(i) Let $\mathcal{A} = \sum_{j=0}^{n-1} A_j(t)\theta^j$ be an invertible linear map between the spaces of solutions. For $0 \le i \le n-1$ we consider the reminder from right-division of $\theta^i\mathcal{A}$ by $L^{(q)}$ in the algebra $(E_p \cap \mathbb{Q}_p[\![t]\!])[\theta]$:

$$\theta^i \mathcal{A} = \sum_{j=0}^{n-1} A_{ij}(t)\theta^j + \mathcal{B}_i \cdot L^{(q)}. \tag{4}$$

Consider matrix Φ with entries $\Phi_{ij} = q^j A_{ij} \in E_p \cap \mathbb{Q}_p[\![t]\!]$. Let $y_0, \dots, y_{n-1}$ be a basis in $Sol(L)$. Consider the constant matrix $\Lambda \in GL_n(\mathbb{Q}_p)$ given by

$$\mathcal{A}(y_j(t^q)) = \sum_{k=0}^{n-1} y_k(t)\lambda_{kj}.$$

Applying θ^i to this identity we find that

$$\sum_{m=0}^{n-1} A_{im}(t) q^m (\theta^m y_j)(t^q) = \sum_{k=0}^{n-1} (\theta^i y_k)(t)\lambda_{kj} \qquad \Leftrightarrow \qquad \Phi(t)U(t^q) = U(t)\Lambda.$$

We obtain that $\Phi(t) = U(t)\Lambda U(t^q)^{-1}$ and therefore it is invertible and satisfies the differential equation (3). □

We would like to note that the entries of Φ and $\mathcal{A}$ in the above proposition were assumed analytic at $t = 0$ in order to have the possibility of multiplication with elements of $Sol(L)$. We could have also assumed that the entries of Φ and $\mathcal{A}$ have a pole of finite order at $t = 0$, that is belong to $E_p \cap \mathbb{Q}_p((t))$.

Remark 11. *In the situation of Proposition 10 the product $L \circ \mathcal{A}$ is right-divisible by $L^{(q)}$ in the algebra $(E_p \cap \mathbb{Q}_p[\![t]\!])[\theta]$. Indeed, let $\mathcal{B} = \sum_{i=0}^{n-1} b_i(t)\theta^i$ be the remainder from division of $-\theta^n \mathcal{A}$ by $L^{(q)}$ on the right. Put $B = (b_i(t)) \in (\mathbb{Q}_p[\![t]\!] \cap E_p)^n$ and consider the vector $C = (A^T)^{-1}B$ where $A = (A_{ij})$ is the matrix defined in* (4)*. Its coordinates satisfy $\sum_{i=0}^{n-1} c_i(t)A_{ij}(t) = b_j(t)$ and therefore*

$$\left(\theta^n + \sum_{i=0}^{n-1} c_i(t)\theta^i\right) \circ \mathcal{A}$$

is right-divisible by $L^{(q)}$. Since $\mathcal{A} : Sol(L^{(q)}) \to Sol(L)$ is invertible, we can conclude that the operator $\tilde{L} = \theta^n + \sum_{i=0}^{n-1} c_i(t)\theta^i$ annihilates all solutions of L. As L and $\tilde{L}$ are monic of the same order, they must be equal. Thus we obtain that $L \circ \mathcal{A} = \tilde{L} \circ \mathcal{A}$ is right-divisible by $L^{(q)}$.

1.5 Existence of Frobenius structure for rigid differential systems

Consider a differential operator

$$L = a_0(t)\left(\frac{d}{dt}\right)^n + a_1(t)\left(\frac{d}{dt}\right)^{n-1} + \ldots + a_{n-1}(t)\frac{d}{dt} + a_n(t)$$

with $a_i \in \mathbb{Q}[t]$ and $a_0 \neq 0$. Let $S = \{t_1, \dots, t_n\} \subset \mathbb{P}^1(\mathbb{C})$ be the singularities of L. This set consists of the roots of $a_0(t)$ and possibly the point at infinity. Let $t_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$ be a regular point and V be the n-dimensional $\mathbb{C}$-vector space of solutions of L near t_0. Consider the monodromy representation

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, t_0) \to GL(V)$$

and assume that it is irreducible. Let $\gamma_1, \dots, \gamma_n \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, t_0)$ be simple loops around $t_1, \dots, t_r$ satisfying the relation $\gamma_1 \cdot \ldots \cdot \gamma_r = I$. Then linear transformations $M_i = \rho(\gamma_i)$ (local monodromies) also satisfy $M_1 \cdot \ldots \cdot M_r = I$. An irreducible tuple $M_1, \dots, M_r$ satisfying the relation $M_1 \cdot \ldots \cdot M_r = I$ is called *rigid* if for any tuple $\tilde{M}_1, \dots, \tilde{M}_r$ such that $\tilde{M}_i = U_i \tilde{M}_i U_i^{-1}$ for all i with some $U_i \in GL(V)$ and $\tilde{M}_1 \cdot \ldots \cdot \tilde{M}_r = I$, there exists a matrix $U \in GL(V)$ such that $M_i = U\tilde{M}_i U^{-1}$ for all i simultaneously. If this condition holds for our tuple of monodromy operators $M_i = \rho(\gamma_i)$ then we say that the *monodromy of L is rigid.*
Let us recall a criterion of rigidity:

Theorem 12 (Katz, [18])**.** *Let $M_1, \dots, M_r \in GL_n(\mathbb{C})$ be an irreducible tuple satisfying the relation $M_1 \cdot \ldots \cdot M_r = I$. Denote $\delta_i = \mathrm{codim}_{\mathbb{C}}\{A \in M_n(\mathbb{C}) : AM_i = M_i A\}$. Then*

(i) $\delta_1 + \ldots + \delta_r \geq 2(n^2 - 1)$,

(i) the tuple is rigid if and only if $\delta_1 + \ldots + \delta_r = 2(n^2 - 1)$.

We now state the theorem on the existence of p-adic Frobenius operators for rigid Fuchsian operators.

Theorem 13 (Vargas-Montoya, [22])**.** *Let $L \in \mathbb{Q}(t)[d/dt]$ and suppose that*

(i) L is Fuchsian,

(ii) exponents of L are rational numbers,

(iii) the monodromy of L is rigid.

Then there exist an integer $h > 0$ such that L has a p-adic Frobenius structure of period h for almost all primes p.

Remark 14. *(i) The construction of the integer $h > 0$ and the set of primes numbers p such that L has a p-adic Frobenius structure are also given in [22, Theorem 3.8].*

(ii)Dwork conjectured in [13] that (i) and (ii) are sufficient for L to have a p-adic Frobenius structure for almost all primes p.

Results similar to Theorem 13 were also obtained by Crew and Esnault-Groechenig circa 2017. One of the advantages of Daniel's approach is that h and the set of bad primes are determined explicitly. Namely, let d be the least common multiple of denominators of all local exponents of L and $P(t)$ be the least common multiple of denominators of the rational coefficients $a_1(t), \ldots, a_n(t)$. Then $h = h_1 h_2$ with $h_1 = \phi(d)$ and h_2 is the dimension of the splitting field of $P(t)$ over $\mathbb{Q}$. Operator L then has a p-adic Frobenius structure of period h for every prime p for which

- all local exponents are p-integral ($\Leftrightarrow p \nmid d$)
- $|a_i(t)|_{\mathcal{G}} \leq 1$ for $i = 1, \ldots, n$
- the difference of any two singularities has p-adic valuation 0

The reference for this is [22, Theorem ?].

2 Exercises

2.1 Amice ring

For every prime number p, the Amice ring is defined as follows

$$\mathcal{A}_p = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n : a_n \in \mathbb{Q}_p, \lim_{n \to -\infty} |a_n|_p = 0 \text{ and } \sup_{n \in \mathbb{Z}} |a_n|_p < \infty \right\}.$$

For every $f = \sum_{n \in \mathbb{Z}} a_n t^n$, we set

$$|f|_{\mathcal{G}} = \sup_{n \in \mathbb{Z}} |a_n|_p.$$

(i) Prove that $|\ |_{\mathcal{G}}$ is a norm. This norm is called the Gauss norm.

(ii) Prove that $\mathcal{A}_p$ is complete with respect to the Gauss norm.

(iii) Prove that $\mathbb{Q}(t) \subset \mathcal{A}_p$ and show that

$$\left| \frac{\sum_i a_i t^i}{\sum_j b_j t^j} \right|_{\mathcal{G}} = \frac{\max_i |a_i|_p}{\max_j |b_j|_p}.$$

Conclude that $E_p \subset \mathcal{A}_p$, where E_p is the p-adic closure of $\mathbb{Q}(t)$ called the field of p-adic analytic elements.

(iv) Show that $\mathcal{A}_p$ is a field.

Remark: Usually, the Amice ring is defined with coefficients in $\mathbb{C}_p$, in which case it is not true that every non-zero element is invertible. It is essential for this exercise that the Gauss norm is discretely valued on $\mathcal{A}_p$.

(v) Show that if $f = \sum_{n\geq 0} a_n z^n \in \mathcal{A}_p$ has radius of convergence greater than 1 then $f \in E_p$.

2.2 Hypergeometric Frobenius structures

A generalized hypergeometric differential operator of order $n \geq 1$ is given by

$$L = (\theta + \beta_1 - 1)(\theta + \beta_2 - 1) \dots (\theta + \beta_n - 1) - t(\theta - \alpha_1) \dots (\theta - \alpha_n), \qquad \theta = t\frac{d}{dt}$$

with some complex numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. This is a Fuchsian operator with singularities at $0, 1, \infty$. The local exponents read

$$\begin{array}{lll} 1-\beta_1, \dots, 1-\beta_n & \text{at} & t=0, \\ \alpha_1, \dots, \alpha_n & \text{at} & t=\infty, \\ 1, 2, \dots, n-1, -1 + \sum_{i=1}^{n} (\beta_i - \alpha_i) & \text{at} & t=1. \end{array}$$

The monodromy representation of L is known to be irreducible if and only if $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j. In his thesis in 1961 Levelt gave a beautiful explicit proof of rigidity of monodromy groups of irreducible hypergeometric monodromy operators (see [6, §1.2]).

(i) Check that an irreducible hypergeometric differential equation satisfies Katz' criterion of rigidity given in Theorem 12.

(ii) Suppose that $\alpha_i, \beta_j \in \mathbb{Q}$ and $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j. Then the hypergeometric operator L satisfies the conditions of Theorem 13. Compute the order of this Frobenius structure and the set of primes for which it exists using the recipe given after Theorem 13.

2.3 p-adic analytic continuation

Let us consider the hypergeometric series

$$\mathfrak{f}(t) = {}_2F_1(1/2, 1/2, 1; t) = \sum_{n\geq 0} \frac{(1/2)_n^2}{n!^2} t^n.$$

Dwork has shown in his "p-adic cycles" paper that, for all $p > 2$, the quotient $\mathfrak{f}(t)/\mathfrak{f}(t^p)$ belongs to E_p. More precisely, he showed that for all $p > 2$ and $s \geq 1$

$$\frac{\mathfrak{f}(t)}{\mathfrak{f}(t^p)} = \frac{\mathfrak{f}_s(t)}{\mathfrak{f}_{s-1}(t^p)} \bmod p^s \quad \text{with} \quad \mathfrak{f}_s(t) = \sum_{n=0}^{p^s-1} \frac{(1/2)_n^2}{n!^2} t^n.$$

(i) Show that the p-adic radius of convergence of $\mathfrak{f}(t)/\mathfrak{f}(t^p)$ is 1 for any $p > 2$.

(ii) Consider the region

$$\mathcal{D} = \{y \in \mathbb{Z}_p : |\mathfrak{f}_1(y)|_p = 1\}$$

and check the following facts:

(a) $\{y \in \mathbb{Z}_p : |y| < 1\} \subset \mathcal{D}$, and if $y \in \mathcal{D}$ then $y^p \in \mathcal{D}$;

(b) for every $s \geq 0$ one has $|\mathfrak{f}_s(y)|_p = 1$ when $y \in \mathcal{D}$;

(c) the sequence of rational functions $\mathfrak{f}_s(y)/\mathfrak{f}_{s-1}(y^p)$ converges uniformly in $\mathcal{D}$, and if we denote the limiting analytic function by $\omega(y) = \lim_{s\to\infty} \mathfrak{f}_s(y)/\mathfrak{f}_{s-1}(y^p)$ then for all $s \geq 1$

$$\sup_{y\in\mathcal{D}} \left| \omega(y) - \frac{\mathfrak{f}_s(y)}{\mathfrak{f}_{s-1}(y^p)} \right| \leq \frac{1}{p^s};$$

(d) $\mathfrak{f}(t)/\mathfrak{f}(t^p)$ is the restriction of $\omega(t)$ to $\{y \in \mathbb{Z}_p : |y|_p < 1\}$.

Remark: The above procedure of analytic continuation allows to evaluate $\omega(y)$ at points $y \in \mathbb{Z}_p^\times$ such that $|\mathfrak{f}(y)|_p = 1$. Dwork also noted that the value $\omega(y_0)$ at a Teichmuller units $y_0 \in \mathbb{Z}_p^\times$, $y_0^{p-1} = 1$ is equal to the p-adic unit root of the elliptic curve $y^2 = x(x-1)(x-\overline{y}_0)$ where $\overline{y}_0$ is the reduction of y_0 modulo p. The condition $|\mathfrak{f}_1(y_0)|_p = 1$ chooses the ordinary elliptic curves in the Legendre family. A vaste generalisation of the above Dwork's congruences along with the evaluation of the respective p-adic analytic element is given in "Dwork crystals II" by Beukers-Vlasenko (see Theorem 3.2 and Remark 4.5).

(iii) Argue that the sequence of rational functions $\mathfrak{f}_s(t)/\mathfrak{f}_{s-1}(t^p)$ converges in the Gauss norm, and hence $\omega(t) \in E_p$

2.4 p-adic Frobenius structure for differential equations of rank 1

(i) Prove that, for any $p > 2$, the differential operator

$$\frac{d}{dt} - \frac{\mathfrak{f}'(t)}{\mathfrak{f}(t)}$$

has a p-adic Frobenius structure of period 1. Here $\mathfrak{f}$ is the hypergeometric function considered in the previous set of exercises.

(ii) Let $L = d/dt - a(t)$ be a differential operator with $a(t) \in \mathbb{Q}(t)$. Prove that if L has a p-adic Frobenius structure for almost all primes p then $a(t) = f'(t)/f(t)$ with $f(t) \in \mathbb{Q}[[t]]$ algebraic over $\mathbb{Q}(t)$. Is the converse true?

Hint: Use the fact that the Grothendieck-Katz p-curvature conjecture holds for operators of rank 1.

(iii) Prove that the differential equation $d/dt - 1$ does not have a p-adic Frobenius structure for any p.

(iv) Let π_p be in $\overline{\mathbb{Q}}$ such that $\pi_p^{p-1} = -p$. Prove that $d/dt - \pi_p$ has a p-adic Frobenius structure.

Remark: A. Pulita in his work *Frobenius structure for rank one p-adic differential equations* gives a characterization of the differential operators of rank 1 having a p-adic Frobenius structure for given p.

3 Algebraicity of G-functions modulo p

Let K be a field and let $f(t)$ be a power series with coefficients in K. We say that $f(t)$ is *algebraic* over $K(t)$ if there exists a nonzero polynomial $P(Y) \in K(t)[Y]$ such that $P(f) = 0$. Otherwise, we say that $f(t)$ is *transcendental* over $K(t)$.

Given a prime number p, $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at prime ideal (p). In other words,

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid (a,b) = 1,\ (p,b) = 1 \right\}.$$

In particular, the elements of the ring $\mathbb{Z}_{(p)}$ can be reduced modulo p and the residue field of $\mathbb{Z}_{(p)}$ is $\mathbb{F}_p$, the field with p elements.

For a power series $f(t) = \sum_{n\geq 0} a_n t^n \in \mathbb{Z}_{(p)}[[t]]$, the reduction modulo p of $f(z)$ is the power series

$$f_{|p}(t) = \sum_{n\geq 0} (a_n \bmod p) z^n \in \mathbb{F}_p[[t]].$$

Definition 15 (Algebraicity modulo p)**.** *Let $f(t)$ be a power series with coefficients in $\mathbb{Q}$. We say that $f(t)$ is algebraic modulo p if:*

1. *we can reduce $f(t)$ modulo p, that is, $f(t)$ belongs to $\mathbb{Z}_{(p)}[[t]]$,*
2. *the reduction of $f(t)$ modulo p is algebraic over $\mathbb{F}_p(t)$, that is, there exits a nonzero polynomial $P(Y) \in \mathbb{F}_p(t)[Y]$ such that $P(f_{|p}) = 0$.*

If $f(t) \in \mathbb{Q}[[t]]$ is algebraic modulo p, the *algebraicity degree* of $f_{|p}(t)$, denoted $deg(f_{|p})$, is the degree of the minimal polynomial of $f_{|p}(t)$ or equivalent

$$deg(f_{|p}) = [\mathbb{F}_p(t)(f_{|p}) : \mathbb{F}_p(t)].$$

3.1 p-adic Frobenius structure implies algebraicity of solutions modulo p

Theorem 16 (Vargas-Montoya,[22])**.** *Let $L \in \mathbb{Q}(t)\left[\frac{d}{dt}\right]$ be a differential operator of order n and $f(t) \in \mathbb{Q}[[t]]$ be a solution of L. If $f(t) \in \mathbb{Z}_{(p)}[[t]]$ and L has a p-adic Frobenius structure of period h then $f(t)$ is algebraic modulo p and $deg(f_{|p}) \leq p^{n^2h}$.*

Proof. Let A be in $M_n(\mathbb{Q}(t))$ the companion matrix of L. Since L has a p-adic Frobenius structure of period h, by Proposition 10, there exists $\mathcal{A} = \sum_{i=0}^{n-1} A_i(t)\left(\frac{d}{dt}\right)^i \in E_p\left[\frac{d}{dt}\right]$ such that, for every solution $y(t)$ of L, the composition $\mathcal{A}(y(t^{p^h}))$ is a solution of L. Consider $V := \{g \in \mathcal{A}_p, Lg = 0\}$. It is clear that V is a $\mathbb{Q}_p$-vector space. Further, the vector $f(t) \in V$ because $f(t) \in \mathbb{Z}_{(p)}$ and $Lf = 0$. We then put

$$\begin{aligned} \psi : V \to & V \\ \vec{y} \mapsto & \mathcal{A}(y(t^{p^h})) \end{aligned}$$

So ψ is a $\mathbb{Q}_p$-linear map. Since $\dim_{\mathbb{Q}_p} V = r \leq n$, from Cayley-Hamilton theorem we get that there are $c_0, \ldots, c_{r-1} \in \mathbb{Q}_p$ such that

$$\psi^r + c_{r-1}\psi^{r-1} + \cdots + c_1\psi + c_0 = 0. \tag{5}$$

Let Z be the E_p vector space generated by the elements of the following set $\{f^{(j)}(t^{p^{ih}}) : j \in \{0, \ldots, n-1\}, i \in \mathbb{N}\}$. From the equality (5) we conclude that Z has dimension less or equals than nr. Since $f(z), \ldots, f(z^{p^{nrh}}) \in Z$, there are $j \leq nr$ and $b_0, \ldots, b_j \in E_p$ such that

$$b_j(t)f(t^{p^{jh}}) + b_{j-1}(t)f(t^{p^{(j-1)h}}) + \cdots + b_0(t)f(t) = 0.$$

Let $b_l(t)$ such that $|b_l(t)| = \max\{|b_0(t)|, \ldots, |b_j(t)|\}$ and define $c_i(t) = b_i(t)/b_l(t)$. Then, for all $i \in \{0, \ldots, j\}$, $|c_i| \leq 1$ and

$$c_j(t)f(t^{p^{jh}}) + c_{j-1}(t)f(t^{p^{(j-1)h}}) + \cdots + c_0(t)f(t) = 0.$$

We set $d_i(t) = \overline{c_i(t)}$, where $\overline{c_i(t)}$ is the reduction of $c_i(t)$ modulo the maximal ideal of ϑ_{E_p}. Then, for all $i \in \{1, \ldots, j\}$, $d_i(t) \in \mathbb{F}_p(z)$,

$$d_j(t)(f_{|p}(t^{p^{jh}})) + d_{j-1}(t)(f_{|p}(t)^{p^{(j-1)h}}) + \cdots + d_0(t)f_{|p}(t) = 0 \tag{6}$$

and $d_0(t), \ldots, d_j(t)$ are not all zero because $1 = \max\{|c_0(t)|, \ldots, |c_j(t)|\}$. As $j \leq nr \leq n^2$ and $\mathbb{F}_p$ has characteristic p, from (6) one gets that $f_{|p}(t)$ is algebraic over $\mathbb{F}_p(t)$ and that $deg(f_{|p}) \leq p^{n^2h}$. $\square$

We now introduce the following sets

$$\mathbf{Algmod} = \{f(t) \in \mathbb{Q}[[t]] \mid f \text{ is algebraic modulo } p \text{ for infinitely many primes } p\}$$

$$\mathbf{Fs} = \{f(t) \in \mathbb{Q}[[t]] \mid f(t) \text{ is solution of a differential operator } L \text{ having Fs for almost all primes } p\}$$

$$\mathbf{Fs}^* = \left\{f(t) \in \mathbb{Q}[[t]] \mid f(t) \in \mathbf{Fs} \text{ and } f(t) \in \mathbb{Z}_{(p)}[[t]] \text{ for infinitely many primes } p\right\}.$$

As a consequence of Theorem 16, we have

$$\mathbf{Fs}^* \subset \mathbf{Algmod}.$$

As an example let us consider $\mathfrak{f}_2(t) = \sum_{n\geq 0} \binom{2n}{n}^2 t^n$. This power series is solution of the differential operator

$$\delta^2 - 16z(\delta + 1/2)^2.$$

According to Exercise 2.2, this differential operator has a p-adic Frobenius structure for all $p > 2$ of period 1. Then, from Theorem 16, $\mathfrak{f}_2(t)$ is algebraic modulo p and $deg(f_{2|p}) \leq p^4$ for all $p > 2$.

Remark 17. *The power series $\mathfrak{f}_2(t)$ is transcendental over $\mathbb{Q}(t)$. Nevertheless, $\mathfrak{f}_{|2}(t)$ is algebraic over $\mathbb{F}_p(t)$ for all $p > 2$.*

The following inclusion will be proven in Theorem 26

$$\mathbf{Fs} \subset \mathbf{G\text{-}functions},$$

where **G-functions** is the class of G-functions introduced by Siegel in 1929. In addition, a famous conjecture due to Bombieri and Dwork suggests that

$$\mathbf{G\text{-}functions} \subset \mathbf{Fs}.$$

Furthermore, Adamczewski and Delaygue recently conjectured that

$$\mathbf{G\text{-}functions}^* \subset \mathbf{Algmod},$$

where **G-functions*** is the set of the power series $f(t) \in \mathbb{Q}[[t]]$ that are G-functions and there exists an infinite set S of prime numbers such that, for all $p \in S$, $f(t) \in \mathbb{Z}_{(p)}[[t]]$.

We are going to see that the Adamczewski-Delaygue's conjecture is true for many of G-functions, namely, *diagonals of algebraic power series* and *hypergeometric* series ${}_nF_{n-1}$ with rational parameters.

3.2 G-functions

We say that $f(t) = \sum_{n\geq 0} a_n t^n \in \mathbb{Q}[[t]]$ is a G-functions if:

(i) there exists a nonzero differential operator $L \in \mathbb{Q}(t)\left[\frac{d}{dt}\right]$ such that $L(f) = 0$,

(ii) there exists $C > 0$ such that $|a_n| < C^{n+1}$ for all $n \geq 0$,

(iii) there exists $D > 0$ and a sequence of integers $D_m > 0$ with $D_m \leq D^{m+1}$ such that $D_m a_n \in \mathbb{Z}$ for all $n \leq m$.

The main examples of G-functions are given by *diagonals of algebraic power series* and *hypergeometric* series ${}_nF_{n-1}$ with rational parameters.

3.2.1 Diagonals

Let K be any field. For every integer $n \geq 1$, we define the diagonalisation operator

$$\Delta_n : K[[t_1, \ldots, t_n]]^{rat} \to K[[t]]$$

$$\sum_{i\in\mathbb{N}^n} a(i_1, \ldots, i_n) t_1^{i_1} \cdots t_n^{i_n} \mapsto \sum_{j\geq 0} a(j, \ldots, j) t^j,$$

where $K[[t_1, \ldots, t_n]]^{rat} = K[[t_1, \ldots, t_n]] \cap K(t_1, \ldots, t_n)$.

Definition 18. *We say that $f(t) \in K[[t]]$ is a diagonal of a rational function if there are an integer $n > 0$ and $F \in K[[t_1, \ldots, t_n]]^{rat}$ such that*

$$\Delta_n(F) = f.$$

We put

$$\mathbf{Diag}_K^{rat} = \{ f(t) \in K[[t]] \mid f(t) \text{ is a diagonal of a rational function } \}.$$

For example, the power series $\mathfrak{f}_2(t) = \sum_{n\geq 0} \binom{2n}{n}^2 t^n$ belongs to $\mathbf{Diag}_{\mathbb{Q}}^{rat}$ because $\Delta_4(R(t_1,\ldots,t_4)) = \mathfrak{f}_2(t)$, whit

$$R(t_1,\ldots,t_4) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} = \sum_{(i_1,i_2,i_3,i_4)\in\mathbb{N}^4} \binom{i_1+i_2}{i_1}\binom{i_3+i_4}{i_3} t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{i_4}.$$

The generating power series of Apéry's numbers

$$\mathfrak{A}(t) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\right)t^n$$

is the diagonal of the following rational function

$$\frac{1}{[1-t_1t_2t_3t_4][(1-t_1)(1-t_4)-t_0(1+t_1)(1+t_2)]}.$$

Theorem 19 (Furstenberg [16])**.** *Let K be a field of characteristic $p>0$. If $f(t) \in \boldsymbol{Diag}^{rat}$ then $f(t)$ is algebraic over $K(t)$.*

This result was extended by Deligne to *diagonal of algebraic power series.* We say that $f(t) \in K[[t]]$ is a *diagonal of an algebraic power series* if there are $n>0$ and $F \in K[[t_1,\ldots,t_n]]^{alg}$ such that $\Delta_n(F) = f$, where $K[[t_1,\ldots,t_n]]^{alg}$ is the set of power series in $K[[t_1,\ldots,t_n]]$ that are algebraic over $K(t_1,\ldots,t_n)$. We then put

$$\mathbf{Diag}_K^{alg} = \{ f(t) \in K[[t]] \mid f(t) \text{ is a diagonal of an algebraic power series}\}.$$

It is clear that $\mathbf{Diag}^{rat} \subset \mathbf{Diag}^{alg}$.

Theorem 20 (Deligne [11])**.** *Let K be a field of characteristic $p>0$. If If $f(t) \in \boldsymbol{Diag}^{alg}$ then $f(t)$ is algebraic over $K(t)$.*

In the following proposition we state some properties of the set $\mathbf{Diag}^{alg}$.

Proposition 21. *The following statements hold.*

1. *For any field K, $\boldsymbol{Diag}_K^{rat} = \boldsymbol{Diag}_K^{alg}$.*
2. *If $f(t) \in \boldsymbol{Diag}_{\mathbb{Q}}^{rat}$ then $f(t)$ is N-integral, that is, there exists $c \in \mathbb{N} > 0$ such that $f(cz) \in \mathbb{Z}[[t]]$.*

Proof. 1. See [12, Theorem 6.2]

2. It is a direct consequence of 1.

□

Thanks to Proposition 21, Theorem 19 and Theorem 20 are equivalent.

Theorem 22. *The following inclusions hold:*

1. $\boldsymbol{Diag}_{\mathbb{Q}}^{rat} \subset$ ***Algmod***,
2. $\boldsymbol{Diag}_{\mathbb{Q}}^{rat} \subset G$***-functions***,
3. $\boldsymbol{Diag}_{\mathbb{Q}}^{rat} \subset \boldsymbol{Fs}^*$.

Proof. Let $f(t)$ be in $\mathbf{Diag}^{rat}_{\mathbb{Q}}$.

1. By 2 of Proposition 21, we get that $f(t) \in \mathbb{Z}_{(p)}[[t]]$ for almost all primes p. By assumption $f = \Delta_n(F)$ with $F \in \mathbb{Q}[[t_1, \dots, t_n]] \cap \mathbb{Q}(t_1, \dots, t_n)$. So, for almost all primers p, $F \in \mathbb{Z}_{(p)}(t_1, \dots, t_n)$. Since $f_{|p} = \Delta_n(F_{|p})$, we conclude that, for almost all primes p, $f_{|p}(t) \in \mathbf{Diag}^{rat}_{\mathbb{F}_p}$. Then, by Theorem 19, we deduce that $f(t) \in$ **Algmod**.

2. It is shown in [10] that $f(t)$ is a solution of a nonzero differential operator $L \in \mathbb{Q}(t)\left[\frac{d}{dt}\right]$. Another proof is given in [20]. The condition (iii) is satisfied because $f(t)$ is globally bounded and the (ii) is also satisfied because the radius of convergence of $f(t)$ is not zero.

3. By [10], we know that $f(t)$ is solution of a Picard-Fuchs operator $L \in \mathbb{Q}(t)\left[\frac{d}{dt}\right]$. Then, according to [19, Theorem 22.1], L is equipped with a p-Frobenius structure for almost all primes p. Finally, 2 of Proposition 21 implies that $f(t) \in \mathbb{Z}_{(p)}[[t]]$ for almost all primes p. Hence $f(t) \in \mathbf{Fs}^*$. □

For a G-function $f(t)$, we let S_f denote the set of primer number p such that $f(t) \in \mathbb{Z}_{(p)}[[t]]$. According to Theorem 22, if $f(t) \in \mathbf{Diag}^{rat}_{\mathbb{Q}}$ then $\mathcal{P} \setminus S_f$ is finite, where $\mathcal{P}$ is the set of prime numbers. Deligne[11] proposed that the behaviour of $deg(f_{|p})$ with respect to $p \in S_f$ is polynomial in p. More precisely,

Deligne's question:(Deligne[11]) Let $f(t)$ be in $\mathbf{Diag}^{rat}_{\mathbb{Q}}$. Is there a constant $c > 0$ such that, for all $p \in S_f$, $deg(f_{|p}) < p^c$?

In the particular case of the power series $\mathfrak{f}_2(t)$, we can take $c = 4$ because we have already seen that $deg(\mathfrak{f}_{2|p}) \leq p^4$ for all $p > 2$. It was in 2013 that Adamczewski and Bell [1] gave an affirmative answer to this question. Further, it is observed that in many examples, Deligne's question has an affirmative answer for G-functions which are not diagonals. For example, the hypergeometric series

$$\mathfrak{h}(t) = \sum_{n \geq 0} \frac{(1/5)_n^2}{(2/7)_n n!} t^n$$

does not belong to $\mathbf{Diag}^{rat}_{\mathbb{Q}}$ because $\mathfrak{h}(t)$ is not N-integral. Nevertheless, $S_{\mathfrak{h}}$ is the set of prime numbers p such that $p = 1 \bmod 35$ and we have $deg(\mathfrak{h}_{|p}) \leq p$ for all $p \in S_{\mathfrak{h}}$.

The fact that Deligne's question has an affirmative answer for many G-functions that are not diagonals led Adamczewski and Delaygue to formulated the following conjecture

Conjecture 23. *(Adamczewski–Delaygue's conjecure) Let $f(t)$ be a G-function such that S_f is infinite. Then:*

(i) $f_{|p}$ is algebraic over $\mathbb{F}_p(t)$ for almost all $p \in S_f$,

(ii) there exists $c > 0$ such that, for all $p \in S_f$ verfying (i), $deg(f_{|p}) < p^c$.

Thanks to Theorem 19 and [1], this conjecture is true when $f(t)$ belongs to $\mathbf{Diag}^{rat}_{\mathbb{Q}}$. Recently, this conjecture was proven for another interesting class of G-functions, namely, hypergeometric series ${}_nF_{n-1}$ with rational parameters.

3.2.2 Hypergeometric series ${}_nF_{n-1}$

Given two vectors $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{n-1}, 1)$ in $(\mathbb{Q} \setminus \mathbb{Z}_{\leq 0})^n$, the hypergeometric series with parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is the power series

$${}_nF_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t) = \sum_{j \geq 0} \mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(j) t^j \text{ with } \mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(j) = \frac{(\alpha_1)_j \cdots (\alpha_n)_j}{(\beta_1)_j \cdots (\beta_{n-1})_j j!},$$

where for a real number x and nonnegative integer j, $(x)_j$ is the Pochhammer symbol, that is, $(x)_0 = 1$ and $(x)_j = x(x+1)\cdots(x+j-1)$ for $j > 0$. We denote by $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the least common multiple of the denominators of $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_{n-1}$ written in lowest form. It is well-known that ${}_nF_{n-1}(\boldsymbol{\alpha},\boldsymbol{\beta};z)$ is a solution of the hypergeometric operator

$$\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta})=\prod_{i=1}^{n}(\delta+\beta_i-1)-z\prod_{i=1}^{n}(\delta+\alpha_i),\ \text{with } \delta=z\frac{d}{dz}.$$

We put

$$\mathbf{HGS}=\{{}_nF_{n-1}(\boldsymbol{\alpha},\boldsymbol{\beta},t)\mid \text{with } n>0 \text{ and } \boldsymbol{\alpha},\boldsymbol{\beta}\in(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^n\}.$$

We have the following inclusion

Theorem 24. ***HGS*** $\subset$ ***G-function***.

Proof. Let $f(t)={}_nF_{n-1}(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ be in **HGS**. We know that $f(t)$ is solution of the differential operator $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta})$. So the condition (i) is verified. Condition (ii) and (iii) follows from [4, Lemma 4.4 Chp I] and [15, Proposition 1.1 Chp VIII] □

In [22], it was proven that the Adamcezwki-Delaygue's conjecture is true for a lot G-functions in **HGS**. To be more precise, we have

Theorem 25. *([22, Theorem 1.2]) Let $f(t)={}_nF_{n-1}(\boldsymbol{\alpha},\boldsymbol{\beta},t)\in$ **HGS** such that $1\leq i,j\leq n$, $\alpha_i-\beta_j\notin\mathbb{Z}$ and let p a prime number such that for all $1\leq i,j\leq n$, $|\alpha_i|_p\leq 1$ and $|\beta_j|_p\leq 1$. If $f(t)\in\mathbb{Z}_{(p)}[[t]]$ then $f_{|p}(t)$ is algebraic over $\mathbb{F}_p(t)$ and $deg(f_{|p})\leq p^{n^2\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})}$, where φ is the Euler's Totient function.*

Proof. We know that $f(t)$ is solution of the differential operator $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta})$. Given that, for all $1\leq i,j\leq n$, $|\alpha_i|_p\leq 1$, $|\beta_j|_p\leq 1$ and $\alpha_i-\beta_j\notin\mathbb{Z}$, it follows from **first session** that $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is equipped with a p-adic Frobenius structure of period $\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})$. By assumption $f(t)\in\mathbb{Z}_{(p)}[[t]]$ and thus, by Theorem 16, $f_{|p}(t)$ is algebraic over $\mathbb{F}_p(t)$ and $deg(f_{|p})\leq p^{n^2\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})}$. □

Thanks to this theorem, the conjecture Adamcezwki-Delaygue's conjecture is true for any $f(t)\in\mathbf{HGS}_{rig}$, where

$$\mathbf{HGS}_{rig}=\{{}_nF_{n-1}(\boldsymbol{\alpha},\boldsymbol{\beta},t)\mid \text{with } n>0,\ \boldsymbol{\alpha},\boldsymbol{\beta}\in(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^n \text{ and } \forall 1\leq i,j\leq n, \alpha_i-\beta_j\notin\mathbb{Z}\}.$$

We finish this section by showing the inclusion **Fs** $\subset$ **G-functions**. In order to prove this inclusion, we recall the notion of the radius of convergence at generic point for a differential equation $L\in\mathbb{Q}(t)\left[\frac{d}{dt}\right]$. Let A be the companion matrix of L and let us consider the sequence of matrices $\{A_s\}_{s\geq 0}$, where A_0 is the identity and $A_1=A$ and $A_{s+1}=\frac{d}{dt}A_s+A_sA$. So, the *radius of convergence of L at the generic point associated to p* is the real number $r_p(L)$ defined as follows

$$\frac{1}{r_p(L)}=\lim_{s\to\infty}\left|\left|\frac{A_s}{s!}\right|\right|_{\mathcal{G},p}^{1/s}.$$

It is not hard to see that $r_p(L)>0$. Moreover, according to Propositions 4.1.2, 4.6.4, 4.7.2 of [9], if L has a p-adic Frobenius structure then $r_p(L)=1$.

Theorem 26. ***Fs*** $\subset$ ***G-functions***.

Proof. Let us take $f(t)\in$ **Fs**. Then $f(t)$ is solution of a differential operator L having p-adic Frobenius structure for almost all primes p. In particular, for almost all primes p, $r_p(L)=1$. Therefore $\prod_p r_p(L)>0$ and, by [4, Theorem C p.3], we conclude that $f(t)$ is a G-function. □

4 Algebraic independence of G-functions

An interesting consequence of the algebraicity modulo p of G-functions is that, in many cases, it allows us to prove the transcendence and algebraic independence of G-functions. We recall that the power series $f_1(t),\dots,f_r(t)\in\mathbb{Q}[[t]]$ are *algebraically independent* over $\mathbb{Q}(t)$, if for any nonzero poylnomial $P(Y_1,\dots,Y_n)\in\mathbb{Q}(t)[Y_1,\dots,Y_n]$, $P(f_1,\dots,f_n)\neq 0$. Otherwise, we say that $f_1(t),\dots,f_r(t)$ are *algebraically dependent* over $\mathbb{Q}(t)$.

It seems that Sharif and Woodcock were the first to use algebraicity modulo p to prove the transcendence of certain G-functions. Indeed, in 1989 they proved [21] that, for all integers $r\geq 2$,

$$\mathfrak{f}_r(z)=\sum_{n\geq 0}\binom{2n}{n}^r t^n$$

is transcendental over $\mathbb{Q}(t)$. Their strategy is based on the following lemma.

Lemma 27. *Let $f(t)\in\mathbb{Z}[[t]]$ be algebraic over $\mathbb{Q}(t)$. Then the sequence $\{deg(f_{|p})\}_{p\in\mathcal{P}}$ is bounded.*

Thus, if $f(t)$ is a power series with coefficients in $\mathbb{Z}$ such that the sequence $\{deg(f_{|p})\}_{p\in\mathcal{P}}$ is not bounded then $f(t)$ is transcendental over $\mathbb{Q}(t)$. So, Sharif and Woodcock showed that, for all integers $r\geq 2$, the sequence $\{deg(\mathfrak{f}_{r|p})\}_{p\in\mathcal{P}}$ is not bounded. For do that, they used the fact $\mathfrak{f}_r(t)$ is p-Lucas for all primes p.

Definition 28. *(p-Lucas congruences) Let $f(t)=\sum_{n\geq 0}a_nt^n$ be a power series in $1+t\mathbb{Q}[[t]]$. We say that $f(t)$ is p-Lucas if:*

(i) $f(t)$ can be reduced modulo p, taht is, $f(t)\in\mathbb{Z}_{(p)}[[t]]$

(ii) the reduction modulo p of $f(t)$ satisfies the equality

$$f_{|p}(t)=\left(\sum_{i=0}^{p-1}(a_n \bmod p)t^i\right)f_{|p}(t)^p \tag{7}$$

For example, Gessel [17] proved that $\mathfrak{A}(t)$ is p-Lucas for all primes p. In problem session we are going to see that, for any $r\geq 1$, $\mathfrak{f}_r(t)$ is p-Lucas for all primes p.

In 1998, Allouche, Gouyou-Beauchamps and Skordev [3] generalized the approach introduced by Sharif and Woodcock by giving a criterion for when a power series that is p-Lucas for all primes p is algebraic over $\mathbb{Q}(t)$. Their result reads as follows

Theorem 29. *Let $f(t)$ be in $\mathbb{Z}[[t]]$. Suppose that $f(t)$ is p-Lucas for almost all primes p. Then $f(t)$ is algebraic over $\mathbb{Q}(t)$ if and only if there is a polynomial $P(t)\in 1+t\mathbb{Q}[t]$ the degree less than or equal to 2 such that $f(t)=P(t)^{-1/2}$.*

As a consequence of this result we can show that $\mathfrak{A}(t)$ is transcendental over $\mathbb{Q}(t)$. Let us suppose by contradiction that $\mathfrak{A}(t)$ is algebraic over $\mathbb{Q}(t)$. Given that $\mathfrak{A}(t)$ is p-Lucas for all primes p then there is a polynomial $P(t)\in 1+t\mathbb{Q}[t]$ the degree less than or equal to 2 such that $\mathfrak{A}(t)=P(t)^{-1/2}$. In particular, $\mathfrak{A}(t)$ is solution of the differential operator

$$P(t)\frac{d}{dt}+\frac{1}{2}P'(t).$$

But, it is well-known that the minimal differential operator for $\mathfrak{A}(t)$ over $\mathbb{Q}(t)$ is given by

$$(1-34t+t^2)t^2\frac{d}{dt^3}+(3-153t+6t^2)t\frac{d}{dt^2}+(1-112t+7t^2)\frac{d}{dt}-5+t.$$

Therefore, $\mathfrak{A}(t)$ is transcendental over $\mathbb{Q}(t)$. We can also use Theorem 29 to prove that $\mathfrak{f}_r(t)$ is transcendental over $\mathbb{Q}(t)$ for $r>1$ given that $\mathfrak{f}_r(t)$ is p-Lucas for all primes p and its minimal differential operator over $\mathbb{Q}(t)$ is given by

$$\delta^r-t(\delta+1/2)^r.$$

Recently, Adamczewski, Bell and Delaygue show how to use the relation (7) to study the algebraic independence of some power series. For this purpose, they introduce the set $\mathcal{L}(S)$, where S is a set of prime numbers.

Definition 30. *Let S be a set of prime numbers. The set $\mathcal{L}(S)$ is the set of power series $f(t) \in 1 + t\mathbb{Q}[[t]]$ such that for every $p \in S$:*

(i) $f(t) \in \mathbb{Z}_{(p)}[[t]]$,

(ii) there are polynomials $A_p(t), B_p(t) \in \mathbb{F}_p(t)$ and a positive integer l such that

$$f_{|p}(t) = \frac{A_p(t)}{B_p(t)} f_{|p}(t)^{p^l},$$

(iii) the degrees of $A_p(t)$ and $B_p(t)$ are less than Cp^l, where C is constant that does not dependent on p.

Remark 31. *Let S be a set of prime numbers. If $f(t) = \sum_{n\geq 0} a_n t^n$ is p-Lucas for all $p \in S$ then $f(t) \in \mathcal{L}(S)$. Indeed, for every $p \in S$, $f(t) \in \mathbb{Z}_{(p)}[[t]]$ and*

$$f_{|p}(t) = A_p(t) f_{|p}(t)^p \text{ with } A_p(t) = \sum_{n=0}^{p-1} a(_n \text{mod} p) t^n.$$

It is clear that the degree of $A_p(t)$ is less than p.

In [2], Adamczewski, Bell and Delaygue prove the following result

Theorem 32. *Let S be an infinite set of prime numbers and let $f_1(t), \ldots, f_r(t) \in \mathcal{L}(S)$. Then, $f_1(t), \ldots, f_r(t)$ are algebraically dependent over $\mathbb{Q}(t)$ if and only if there are $a_1, \ldots, a_r \in \mathbb{Z}$ not all zero such that*

$$f_1(t)^{a_1} \cdots f_r(t)^{a_r} \in \mathbb{Q}(t).$$

By applying this theorem, the authors prove that all elements of the set $\{\mathfrak{f}_r(t)\}_{r\geq 2}$ are algebraically independent over $\mathbb{C}(t)$ (see [2, Theorem 2.1]). In the same work they also show that many G-functions are p-Lucas for infinitely many primes p. In order to do that, they study in detail the p-adic valuation of the coefficients of the hypergeometric series and the power series obtained as specialization of factorial ratios. Furthermore, in [23] the question of determining when a power series belongs to $\mathcal{L}(\mathcal{S})$ for an infinite set $\mathcal{S}$ of prime numbers is addressed from the point of view of the p-adic Frobenius structure.
Before stating the main result of [23], we recall that $D \in \mathbb{Q}(t)[\frac{d}{dt}]$ is *MUM at zero* if zero is a regular singular point of D and the exponents at zero of D are all zero. Finally, we recall that Cartier operator associated to p is the $\mathbb{Q}$-linear map $\Lambda_p : \mathbb{Q}[[t]] \to \mathbb{Q}[[t]]$ defined as follows $\Lambda_p(\sum_{n\geq 0} a_n t^n) = \sum_{n\geq 0} a_{np} t^n$.

Theorem 33. *Let $f(t) = \sum_{n\geq 0} a_n t^n$ be in $\boldsymbol{Fs}^*$ and let $\mathcal{S}_f$ be the set of prime numbers p such that $f(t) \in \mathbb{Z}_{(p)}[[t]]$. Suppose that $f(t)$ is solution of a differential operator $D \in \mathbb{Q}(t)[\frac{d}{dt}]$ that is MUM at zero. Then:*

1. *there exist a constant $C > 0$ and a set $\mathcal{S}' \subset \mathcal{S}_f$ such that $\mathcal{S}_f \setminus \mathcal{S}'$ is finite and, for all $p \in \mathcal{S}'$,*

$$f_{|p}(t) = \frac{A_p(t)}{B_p(t)} f_{|p}(t)^{p^l},$$

where $A_p(t), B_p(t)$ belong to $\mathbb{F}_p[t]$ and their degrees are bounded by Cp^{2l}.

2. *Moreover, if for all $p \in \mathcal{S}_f$, $\Lambda_p(f_{|p}) = f_{|p}$ then $f(t) \in \mathcal{L}(\mathcal{S}')$, where $\mathcal{S}' \subset \mathcal{S}_f$ and $\mathcal{S}_f \setminus \mathcal{S}'$ is finite.*

Theorem 33 is used in [23] to prove that a big class of G-functions are in $\mathcal{L}(\mathcal{S})$, where $\mathcal{P} \setminus \mathcal{S}$ is finite. Mainly, the author uses this theorem to show that the amongst the 400 power series appearing in [5] there are 242 that belong to $\mathcal{L}(\mathcal{S})$. Actually, according to some standard conjectures, it is expected that all power series in [5] belong to $\mathbf{Fs}^*$.

In [23] it is also shown that some hypergeometric series do not belong to $\mathcal{L}(\mathcal{S})$ for any infinite set $\mathcal{S}$ of prime numbers. For any $r \geq 1$, we consider the hypergeometric series

$$\mathfrak{g}_r(t) = \sum_{n\geq 0} \frac{-1}{2n-1} \binom{2n}{n}^r t^n \in 1 + t\mathbb{Z}[[t]].$$

It is easy to check that $\mathfrak{g}_r(t)$ is not p-Lucas for any $p > 2$. Further, in [23] it is shown that if $\mathcal{S}$ is an infinite set of prime numbers then $\mathfrak{g}_2(t) \notin \mathcal{L}(\mathcal{S})$. The arguments given in [23] lead us to think that the same situation is true for $\mathfrak{g}_r(t)$ with $r > 2$. However, for any $r \geq 1$ the hypergeometric series $\mathfrak{g}_r(t)$ satisfies the assumptions of Theorem 33 because $\mathfrak{g}_r(t)$ is solution of the heypergeometric operator

$$\delta^r - t(\delta - 1/2)(\delta + 1/2)^{r-1}.$$

It is clear that this operator is MUM at zero and, according to the **first session**, has a p-adic Frobenius structure for all $p > 2$.
Consequently, we are not able to apply Theorem 32 to the set $\{\mathfrak{g}_r(t)\}_{r\geq 2}$. So, this raises the problem of giving an algebraic independence criterion for the power series that verify the assumptions of Theorem 33. Even if we do not yet have of such a criterion, it is shown in [23] the following results.

Theorem 34. *(i) All elements of the set $\{\mathfrak{g}_r(t)\}_{r\geq 2}$ are algebraically independent over $\mathbb{Q}(t)$.*

(ii) The power series $\mathfrak{g}_2(t)$ and $\mathfrak{A}(t)$ are algebraically independent over $\mathbb{Q}(t)$.

5 Exercises

The goal of this problem session is to prove that $\mathfrak{f}_r(t) = \sum_{n\geq 0} \binom{2n}{n}^r t^n$ is transcendental over $\mathbb{Q}(t)$ following the approach given by Sharif and Woodcock.

5.1 Diagonals

(i) Given two power series $f(t) = \sum_{n\geq 0} t^n$ and $g(t) = \sum_{n\geq 0} = \sum_{n\geq 0} b_n t^n$, the *Hadamard product* of $f(t)$ and $g(t)$ is defined as follows:

$$f(t) \star g(t) = \sum_{n\geq 0} a_n b_n t^n.$$

Prove that if $f(t)$ and $g(t)$ belong to $\mathbf{Diag}_K^{rat}$ then $f(t) \star g(t)$ belongs to $\mathbf{Diag}_K^{rat}$. Conclude that, for all $r \geq 1$, $\mathfrak{f}_r(t)$ belongs to $\mathbf{Diag}_{\mathbb{Q}}^{rat}$.

5.2 Algebraicity modulo p

According to Theorems 13 and 16, $\mathfrak{f}_{|p} r(t)$ is algrebaic modulo p for all $p > 2$ and $deg(\mathfrak{f}_{r|p}) \leq p^{r^2}$. In this exercise we are going to prove that $\mathfrak{f}_r(t)$ is p-Lucas for all primes p.

(ii) Lucas' Theorem. Let p be a prime number and $n = \sum_{i=0}^{s} n_i p^i$, $m = \sum_{i=0}^{s} m_i p^i$ be the p-adic expansion expansion of $n, m \in \mathbb{N}$. Prove that

$$\binom{n}{m} = \prod_{i=0}^{s} \binom{n_i}{m_i} \bmod p.$$

(iii) Prove that

$$\binom{ap+s}{bp+t} = \binom{a}{b}\binom{r}{s} \bmod p$$

for any $a, b \in \mathbb{N}$ and any $0 \leq t, s < p$.

(iv) Let $p > 2$. Prove that

$$\binom{2(np+m)}{np+m}^r \equiv \begin{cases} \binom{2n}{n}^r \binom{2m}{m}^r \bmod p \text{ si } m \in \{0, 1 \ldots, (p-1)/2\} \\ \\ 0 \bmod p \text{ si } m \in \{(p+1)/2, \ldots, p-1\}. \end{cases}$$

(v) Conclude that $\mathfrak{f}_r(t)$ is p-Lucas for all primes p and that $P_r(\mathfrak{f}_{r|p}) = 0$ where

$$P_r(Y) = Y^{p-1} - \sum_{n=0}^{(p-1)/2} \left(\binom{2n}{n} \bmod p\right) t^n$$

5.3 Transcendence

(vi) Prove Lemma 27.

(vii) Let $N > 0$ be a natural number. Then there exist infinitely many primes p such that if a divides $p-1$ then $a = 1, 2$ or $a > N$.

(viii) Let $N > 0$ be a natural number and let $r \geq 2$. Then there exists a primer number p such that $deg(\mathfrak{f}_{r|p}) > N$.

Hint: Let $a = [K : \mathbb{F}_p(t)]$, where K is the splitting field of $P_r(Y)$. Prove that a divides $p-1$ and that $deg(\mathfrak{f}_{r|p})$ divides a.

In the previous approach, the fact that $\mathfrak{f}_r(t)$ is p-Lucas for all $p > 2$ is crucial for proving the transcendence of $\mathfrak{f}_r(t)$, $r \geq 2$. However, in some cases we can prove transcendence without assuming p-Lucas condition. For every $r \geq 1$, we consider the hypergeometric series

$$\mathfrak{g}_r(t) = \sum_{n \geq 0} \frac{-1}{2n-1} \binom{2n}{n}^r t^n \in 1 + t\mathbb{Z}[[t]].$$

(ix) Prove that $\mathfrak{g}_1(t)$ is algebraic.

(x) Prove that, for all $r \geq 1$, $\mathfrak{g}_r(t)$ is not p-Lucas for any $p > 2$.

(xi) Prove that, for all $p > 2$, $\mathfrak{g}_r(t) = A_p(z)\mathfrak{f}_r(t)^p$, where $A_p(t) \in \mathbb{F}_p[t]$ has degree less than p.

(xii) Prove that, for all $p > 2$, $deg(\mathfrak{g}_{r|p}) = deg(\mathfrak{f}_{r|p})$.

(xiii) Conclude that $\mathfrak{g}_r(t)$ is transcendental over $\mathbb{Q}(t)$ for $r > 1$.

References

[1] B. Adamczewski, and J. P. Bell, *Diagonalization and rationalization of algebraic Laurent series,* Ann. Sci. Éc. Norm. Supér **46** (2013), 963–1004.

[2] B. Adamczewski, J. P. Bell, and E. Delaygue, *Algebraic independence of G-functions and congruences "à la Lucas",* Ann. Sci. Éc. Norm. Supér **52** (2019), 515–559.

[3] J-P. Allouche, D.Gouyou-Beauchamps, G. Skordev, *Transcendence of binomial and Lucas' formal power series.* J. Algebra **210** (1998), 577–592.

[4] Y. André, G-functions and geometry, Aspects of Mathematics **E13**, Friedr. Vieweg & Sohn, Braunschweig, 1989.

[5] G. Almkvist, C. Van Eckenvort, D. Van Straten, and W. Zudilin. *Tables of Calabi-Yau equations,* prepprint 2010, arXiv:0507430 130 pp.

[6] F. Beukers, Hypergeometric functions of one variable, notes from MRI springschool 1999 held in Groningen, `https://webspace.science.uu.nl/~beuke106/springschool99.pdf`

[7] F. Beukers, Gauss' hypergeometric function, Progress in Mathematics 260 (2007), 23–42

[8] A. Haefliger, *Local theory of meromorphic connections in dimension 1 (Fuchs theory)*, Chapter III in Borel et al, Algebraic D-modules.

[9] G. Christol, Modules différentiels et équations différentielles p-adiques, Queen's Papers in Pure and Applied Mathematics, **66**, Queen's University, Kingston, 1983.

[10] G. Christol, *Diagonales de fractions rationnelles et équations de Picard-Fuchs,* Group de travail d'analyse ultramétrique **12** (1984/85), Exp No 13, 12 pp.

[11] P. Deligne, *Intégration sur un cycle évanescent,* Invent. Math **76** (1983), 129-143.

[12] J. Denef, and L. Lipshitz, *Algebraic power series and diagonals,* J. Number Theory **26** (1987) 46–67.

[13] B.M Dwork, *On p-adic differential equations I. The Frobenius structure of differential equations,* Bull. Soc. Math. France **39-40** (1974), 27–37.

[14] B.Dwork, On the uniqueness of Frobenius operator on differential equations, Advanced Studies in Pure Mathematics 17, 1989 (Algebraic Number Theory - in honor of K. Iwasawa), 89–96.

[15] B. Dwork, G. Gerotto, And F. Sullivan, A introduction to G-functions, Annals of Mathematics Studies **133**, Princeton University Press, 1994.

[16] H. Furstenberg, *Algebraic functions over finite fields,* J. Algebra **7** (1967) 271–277.

[17] I. Gessel, *Some congruences for Apéry numbers,* J. Number Theory **14** (1982), 362–368.

[18] N.M. Katz, Rigid Local Systems, Annals of Math.Studies 139, Princeton 1996.

[19] K. Kedlaya, p-adic differential equations, Cambridge studies in advanced mathematics **125**, Cambridge University Press, 2010.

[20] L. Lipshitz, *The diagonal of a D-finite power series is D-finite*, J. Algebra, **113** (1988), 373-378.

[21] H. Sharif and C. F. Woodcock *On the transcendence of certain series,* J. Algebra **121** (1989), 364–369.

[22] D. Vargas-Montoya, *Algébricité modulo p, séries hypergéométriques et structure de Frobenius forte,* Bull. Soc. Math. France **149** (2021), 439–477

[23] D. Vargas-Montoya, *Monodromie unipotente maximale, congruences "à la Lucas" et indépendance algébrique,* Trans. Amer. Math. Soc.**377**(2024), 167–202.